

# Fractional Talbot effect in phase space: A compact summation formula

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**Abstract:** A phase space description of the fractional Talbot effect, occurring in a one-dimensional Fresnel diffraction from a periodic grating, is presented. Using the phase space formalism a compact summation formula for the Wigner function at rational multiples of the Talbot distance is derived. The summation formula shows that the fractional Talbot image in the phase space is generated by a finite sum of spatially displaced Wigner functions of the source field.

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It is the purpose of this communication to investigate a one dimensional Fresnel diffraction from a periodic grating and the corresponding fractional Talbot effect [1] using the Wigner phase space distribution function.

We use the following definition of the Wigner distribution function [2] for the field amplitude  $E(x)$  along the axis  $x$  perpendicular to the propagation direction:

$$W(x, u) = \frac{1}{2\pi} \int dy E^*(x + y/2) e^{iuy} E(x - y/2), \quad (1)$$

where  $u$  has the meaning of a spatial frequency. We consider a monochromatic plane wave characterized by its electric field  $\mathcal{E}(x; z) = e^{ikz} E(x; z)$  propagated paraxially along the  $z$ -axis. At  $z = 0$ , where the field starts to propagate, there is an infinite one-dimensional periodic grating with transmittance  $t(x) = t(x + a)$ . The electric field amplitude after the passage through the grating can be expanded into the Fourier series:

$$E(x; 0) = t(x)E_0 = \sum_{n=-\infty}^{\infty} t_n e^{2\pi i n x/a}, \quad (2)$$

where  $a$  is the period of the grating and  $E_0$  is a constant amplitude of the incident wave. The Wigner distribution function of this source field is given by:

$$\begin{aligned} W(x, u; 0) &= \sum_{n=-\infty}^{\infty} |t_n|^2 \delta(u - nu_0) \\ &+ \sum_{n \neq n'} t_n t_{n'}^* \exp[2\pi i(n - n')x/a] \delta(u - \frac{n + n'}{2}u_0). \end{aligned} \quad (3)$$

Two different types of contributions can be distinguished in the above formula. The first one is a set of parallel positive density stripes separated by the spatial frequency  $u_0 = 2\pi/a$ , given by the reciprocal grid spacing. Each of them is generated by a separate Fourier component of the source field. Coherence between the Fourier components results in nonpositive oscillatory terms of the Wigner function, located precisely in the middle between the contributing frequencies. The nonpositive interference terms are a consequence of the linear superposition principle and the bilinear character of the Wigner function.

The paraxial propagation of the field through a distance  $z$  is described in the phase space as the following simple transformation of the Wigner distribution function [2]:

$$W(x, u; z) = W(x - \frac{\lambda z}{2\pi}u, u; 0). \quad (4)$$

This formula applied to the Wigner function of the periodic field gives the following result:

$$\begin{aligned} W(x, u; z) &= \sum_{n=-\infty}^{\infty} |t_n|^2 \delta(u - nu_0) \\ &+ \sum_{n \neq n'} t_n t_{n'}^* \exp[2\pi i(n - n')x/a - 2\pi i(\theta_n - \theta_{n'})] \delta(u - \frac{n + n'}{2}u_0). \end{aligned} \quad (5)$$

In the course of propagation the interference terms acquire additional phase shifts given by

$$\theta_n = \frac{z}{z_T} n^2, \quad (6)$$

where  $z_T = a^2/2\lambda$  is the Talbot distance. It is straightforward to see that for integer multiples of the Talbot distance the original Wigner function of the input field is reproduced.

At intermediate distances, the phase shifts  $\theta_n$  play a nontrivial role and the structure of the observed Fresnel images becomes more complex. Nevertheless, they exhibit an interesting regular behavior at rational multiples of the Talbot distance. We will now discuss this effect in terms of the Wigner distribution function. Let us denote  $z/z_T = p/q$ , where  $p$  and  $q$  are coprime integers. The main complication in Eq. (5) are phase factors  $\theta_n$  which depend quadratically on  $n$ . We will simplify it with the help of an observation used in the studies of quantum wave packets dynamics [3]: the exponent  $\exp(-2\pi i\theta_n)$  is periodic in  $n$  with the period  $l = q/4$  if  $q$  is a multiple of 4 and  $l = q$  otherwise. The quadratic phase factor can therefore be represented as a finite Fourier sum:

$$\exp(-2\pi i\theta_n) = \sum_{s=0}^{l-1} a_s \exp(-2\pi i s n / l) \quad (7)$$

with certain coefficients  $a_s$ , which have been analyzed in detail in Ref. [3]. Substitution of the above expression yields:

$$\begin{aligned} W(x, u; pz_T/q) &= \sum_{s,s'=0}^{l-1} a_s a_{s'}^* \sum_{n,n'=-\infty}^{\infty} t_n t_{n'}^* \exp\left[\frac{2\pi i n}{a} \left(x - \frac{sa}{l}\right)\right] \\ &\times \exp\left[-\frac{2\pi i n'}{a} \left(x - \frac{s'a}{l}\right)\right] \delta\left(u - \frac{n+n'}{2} u_0\right). \end{aligned} \quad (8)$$

The interference terms of the Wigner function can now be interpreted as generated by pairs of Fourier components of the source field shifted in the position space by rational fractions  $sa/l$ , where  $s = 0, 1, \dots, l-1$ . A simple rearrangement of the exponent arguments allows one to represent the sum over  $n$  and  $n'$  solely in terms of the source field Wigner function:

$$W(x, u; pz_T/q) = \sum_{s,s'=0}^{l-1} a_s a_{s'}^* \exp[iu(s' - s)a/l] W\left(x - \frac{(s+s')a}{2l}, u; 0\right). \quad (9)$$

This is a compact summation formula for the Wigner function at rational multiples of the Talbot distance. It shows, quite surprisingly, that  $W(x, u; pz_T/q)$  is simply given by a *finite* sum of spatially displaced Wigner functions of the source field, with some phase factors.

Integration of the derived expression for the Wigner function over  $u$  yields the known formula for the field intensity distribution in the observation plane:

$$\begin{aligned} |E(x; pz_T/q)|^2 &= \int du W(x, u; pz_T/q) \\ &= \left| \sum_{s=0}^{l-1} a_s t(x - sa/l; 0) \right|^2, \end{aligned} \quad (10)$$

which shows that the observed Fresnel image can be represented as generated by a finite sum of shifted source field amplitudes [4]. A detailed discussion of fractional Talbot images can be found in Ref. [5].

We have discussed a one dimensional Fresnel diffraction from a periodic grating using the phase space formalism. We have seen in this picture, that the whole propagation of the field is encoded in the interference terms of the Wigner function, generated

by coherence between Fourier components of the source wave. The variety of the Fresnel images is simply a result of a complicated interplay between phases of these interference terms. The summation formula derived in this paper demonstrates that the fractional Talbot effect can be also understood as an interference between components shifted in the position space by fractions of the pattern period.

Let us finally note a close analogy between the Talbot effect and the dynamics of various nonlinear quantum systems. Dynamical phase factors with a quadratic dependence on the summation index are a common feature of many quantum mechanical problems. These phase factors can be easily revealed in the equation of motion for the quantum Wigner function in the  $(x, p)$  phase space:

$$W_\psi(x, p; t) = \sum_{n, n'} \psi_n^* \psi_{n'} \exp[i(E_n - E_{n'})t/\hbar] W_{nn'}(x, p), \quad (11)$$

where  $\psi_n$  are the amplitudes of the initial wave function  $\psi(x; 0)$  projected onto the energy eigenstate  $\varphi_n(x)$ , and where

$$W_{nn'}(x, p) = \frac{1}{2\pi\hbar} \int dy \varphi_n^*(x + y/2) e^{ipy/\hbar} \varphi_{n'}(x - y/2) \quad (12)$$

are the cross Wigner functions of the eigenstates. For many systems, their eigenenergies  $E_n$  are given by a quadratic polynomial of  $n$ . This can be either an approximate dependence, as it is for the Rydberg electron wave packets [6], or a strict one, as it happens for an electromagnetic field mode in the Kerr medium [7,8] and a particle in the infinite square well. In all these cases, the state of the system turns out to be highly regular for times equal to rational multiples of the characteristic revival time, at which the system is again in its initial state. This regularity consists in a formation of a finite superposition of copies of the initial state generated by a simple transformation. For the wave packets dynamics, the components are spatially shifted [3]; for the Kerr medium, the components are rotated in the phase space around its origin [9].

In particular, a close similarity exists between the fractional revivals in the infinite square well [10] and the Talbot effect. The dynamics of the former system is equivalent to an evolution in a free space of an infinite periodic sequence of displaced initial wave functions [11,12]. Furthermore, the propagation of electromagnetic fields in the paraxial approximation is governed by the same wave equation as the quantum time evolution. Therefore, Fresnel diffraction of classical electromagnetic waves on periodic gratings is a strict counterpart of the quantum dynamics of a confined particle.

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