

# Maxwell's Demon at work: Two types of Bose condensate fluctuations in power-law traps

Siegfried Grossmann and Martin Holthaus

Fachbereich Physik der Philipps-Universität, Renthof 6, D-35032 Marburg,  
Germany

holthaus@stat.physik.uni-marburg.de

**Abstract:** After discussing the key idea underlying the Maxwell's Demon ensemble, we employ this idea for calculating fluctuations of ideal Bose gas condensates in traps with power-law single-particle energy spectra. Two essentially different cases have to be distinguished. If the heat capacity remains continuous at the condensation point in the large- $N$ -limit, the fluctuations of the number of condensate particles vanish linearly with temperature, independent of the trap characteristics. If the heat capacity becomes discontinuous, the fluctuations vanish algebraically with temperature, with an exponent determined by the trap. Our results are based on an integral representation that yields the solution to both the canonical and the microcanonical fluctuation problem in a singularly transparent manner.

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According to grand canonical statistics, the root-mean-square fluctuations  $\delta N_\nu$  of the occupation numbers  $N_\nu$  of an ideal Bose gases's  $\nu$ -th single-particle state are given by [1,2]

$$(\delta N_\nu)^2 = N_\nu(N_\nu + 1) . \tag{1}$$

This expression follows without any approximation from the grand canonical approach, but it faces a severe problem when applied to the fluctuations  $\delta N_0$  of the ground state occupation number  $N_0$  of *isolated* Bose gases: if the temperature approaches zero, all  $N$  particles of an isolated Bose gas occupy the ground state, so that the actual fluctuations vanish, whereas Eq. (1) predicts fluctuations  $\delta N_0$  of the order  $N$ . This seems to be one of the most important examples where the different statistical ensembles can not be regarded as equivalent. When computing low-temperature fluctuations of the ground state occupation number for isolated Bose gases, one therefore has to give up the convenient grand canonical point of view, and to resort to a microcanonical treatment.

Although this problem had been well recognized and discussed some time ago [3,4], tools for computing the microcanonical fluctuations  $\delta N_0$  have been developed only recently [5], spurred by the progress in preparing Bose–Einstein condensates of alkali atoms in magnetic traps [6,7,8]. A particularly instructive model system for illustrating the microcanonical approach to fluctuations  $\delta N_0$  is provided by  $N$  ideal Bosons trapped in a one-dimensional harmonic potential. Since quasi one-dimensional harmonic trapping potentials can be realized as limiting cases of strongly anisotropic three-dimensional traps [9,10], this system is not merely of academic interest. The value of the model lies in the fact that it allows one to map the problem of evaluating the microcanonical statistics to problems also arising in analytic number theory, because the number of microstates accessible at some excitation energy  $E$  equals the number of partitions of the integer  $n = E/(\hbar\omega)$  into no more than  $N$  summands, with  $\omega$  being the oscillator frequency. Using the appropriate asymptotic formulae from partition theory, one finds that the microcanonical fluctuations  $\delta N_0$  for this model system vanish linearly with temperature  $T$  [11,12]:

$$\delta N_0 \approx \frac{\pi}{\sqrt{6}} \frac{k_B T}{\hbar \omega} \quad \text{for} \quad T \ll T_0^{(1)} \equiv \frac{\hbar \omega}{k_B} \frac{N}{\ln N}, \quad (2)$$

where  $k_B$  is the Boltzmann constant, and  $T_0^{(1)}$  denotes the temperature below which the ground state occupation becomes significant. As illustrated in Fig. 1, which compares the relative microcanonical fluctuations  $\delta N_0/N$  to the corresponding grand canonical fluctuations and to the approximation (2), for  $N = 10^6$  particles, this approximation is quite good indeed. The very same result (2) has also been obtained by Wilkens [13] within a *canonical* approach, that is, for a trap in contact with a heat bath.

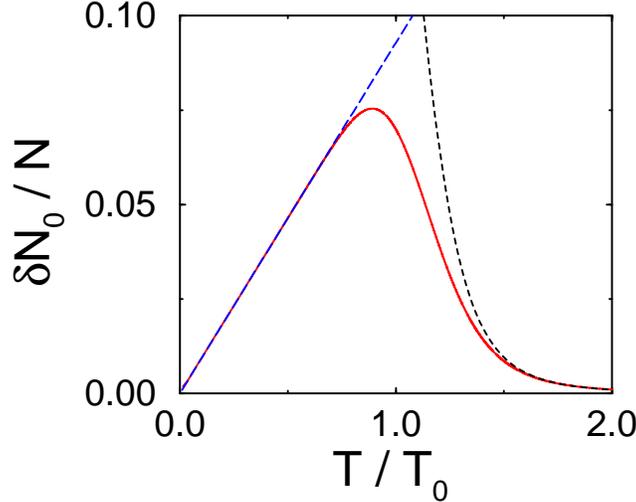


Fig. 1. Full red line: microcanonical fluctuations  $\delta N_0/N$  for a system of  $N = 10^6$  ideal Bose particles trapped by a one-dimensional harmonic potential [11]. The temperature  $T_0 \equiv T_0^{(1)}$  denotes the characteristic temperature below which the ground state occupation becomes significant, see Eq. (2). Black short-dashed line: grand canonical fluctuations for the same system. Blue dashed line: low-temperature approximation provided by Eq. (2).

How can one generalize this finding to other trap types? A rather interesting suggestion has been made by Navez *et al.* [14]. Denoting, for an ideal  $N$ -particle Bose gas in some arbitrary trap, the number of microstates with *exactly*  $N_{\text{ex}}$  excited particles as  $\Phi(N_{\text{ex}}|E)$ , so that the total number of microstates accessible at the given energy  $E$  reads  $\Omega(E|N) = \sum_{N_{\text{ex}}=0}^N \Phi(N_{\text{ex}}|E)$ , these authors consider the generating function

$$\Upsilon(z, E) = \sum_{N_{\text{ex}}=0}^{\infty} z^{N_{\text{ex}}} \Phi(N_{\text{ex}}|E). \quad (3)$$

This function involves  $\Phi(N_{\text{ex}}|E)$  even for  $N_{\text{ex}} > N$ , which appears to be unphysical: after all, the excitation energy  $E$  can not be distributed over more than the  $N$  particles. However, *provided* the microcanonical distributions for finding  $N_{\text{ex}}$  out of  $N$  particles in an excited trap state,

$$p_{\text{ex}}(N_{\text{ex}}|E) = \frac{\Phi(N_{\text{ex}}|E)}{\Omega(E|N)}, \quad N_{\text{ex}} = 1, 2, \dots, N, \quad (4)$$

are strongly peaked around some value  $\bar{N}_{\text{ex}} \ll N$ , which should be the case for temperatures well below the onset of Bose-Einstein condensation, we will have  $\Phi(N_{\text{ex}}|E)/\Omega(E|N) \approx 0$  for  $N_{\text{ex}} > N$ . In that case the generating function (3) would be quite useful, since one could obtain the microcanonical expectation value  $\langle N_0 \rangle$  for the ground state occupation number, and its fluctuation, from

$$N - \langle N_0 \rangle = z \frac{\partial}{\partial z} \ln \Upsilon(z, E) \Big|_{z=1} \quad \text{and} \quad (\delta N_0)^2 = \left( z \frac{\partial}{\partial z} \right)^2 \ln \Upsilon(z, E) \Big|_{z=1},$$

respectively [14]. The proviso can be formulated in more intuitive terms: the required well-peakedness of the distributions (4) means that those microstates where the energy  $E$  is actually spread out over all  $N$  particles carry only negligible statistical weight, so that the overwhelming majority of all microstates leaves a fraction of the particles in the ground state, forming the Bose condensate. Then the restriction on the number of microstates caused by the fact that there is only a finite number  $N$  of particles becomes meaningless, so that, loosely speaking, “the system has no chance to know how many particles the condensate consists of”. But if this is the case, i.e., if the system’s properties become insensitive to the actual number of particles contained in the condensate, then one can act as if the condensate particles constituted an *infinite* reservoir. Thus, the generating function (3) may be regarded as the partition function of a rather unusual ensemble, consisting of the excited-states subsystems of Bose gases that exchange particles with the ground state “reservoirs” without exchanging energy. Since such an exchange process, if performed by hand, requires a genius who is able to separate the hot, excited particles from the cold ones in the ground state, this new ensemble has been called the “Maxwell’s Demon ensemble” [14].

But can we rely on Maxwell’s Demon, that is, does the proviso hold? This question needs to be answered first. A strong argument in favour of the Maxwell’s Demon ensemble has already been provided by the approximation (2) to the low-temperature fluctuations for a Bose gas in a one-dimensional oscillator potential: these fluctuations are *independent of the total particle number  $N$* , as they should be if the system really has no knowledge of the number of condensate particles, and thus of  $N$ . The awe-inspiring agreement with the actual microcanonical fluctuations depicted in Fig. 1 leaves no doubt that this approximation *is* reliable. To further substantiate the new ensemble, we also consider the microcanonical fluctuations  $\delta N_0$  for an ideal Bose gas trapped by a three-dimensional isotropic harmonic oscillator potential [15]. The numbers  $\Omega(E|N)$  of microstates for some given excitation energy  $E = n\hbar\omega$  can then be obtained from the canonical  $N$ -particle partition function

$$Z_N(\beta) = \sum_{n=0}^{\infty} e^{-n\beta\hbar\omega} \Omega(n\hbar\omega|N), \quad (5)$$

which, in turn, can be calculated numerically with the help of the recursion relation [16,17,18]

$$Z_N(\beta) = \frac{1}{N} \sum_{k=1}^N Z_1(k\beta) Z_{N-k}(\beta). \quad (6)$$

As usual,  $\beta = 1/(k_B T)$  denotes the inverse temperature. By means of numerical saddle-point inversions of Eq. (5), we compute the desired numbers  $\Omega(E|N_{\text{ex}})$  for  $N_{\text{ex}}$  ranging from 1 to  $N$  [15], and get the differences

$$\Phi(N_{\text{ex}}|E) = \Omega(E|N_{\text{ex}}) - \Omega(E|N_{\text{ex}} - 1) \quad (7)$$

that determine the microcanonical distributions (4). Some of these distributions are displayed in Fig. 2, for  $N = 1000$  and several “low” temperatures. What we find is exactly what is needed for Maxwell’s Demon: the distributions are well peaked for temperatures below the onset of condensation, and remarkably close to Gaussians [12]. It is then no surprise that the corresponding microcanonical low-temperature fluctuations  $\delta N_0$ , obtained from the widths of these distributions, are — as long as a condensate

exists! — once again independent of  $N$ , as exemplified in Fig. 3 for  $N = 200, 500,$  and  $1000$ . As discussed above, it is precisely this  $N$ -independence, expressed mathematically by the appearance of the upper summation bound “ $\infty$ ” rather than “ $N$ ” in Eq. (3), that lies at the bottom of the Maxwell’s Demon ensemble. But whereas this  $N$ -independence is, by construction, *put into* this ensemble, it has *come out* here as the result of a truly microcanonical calculation [12,15] that works with the actual  $N$ , not with  $\infty$ .

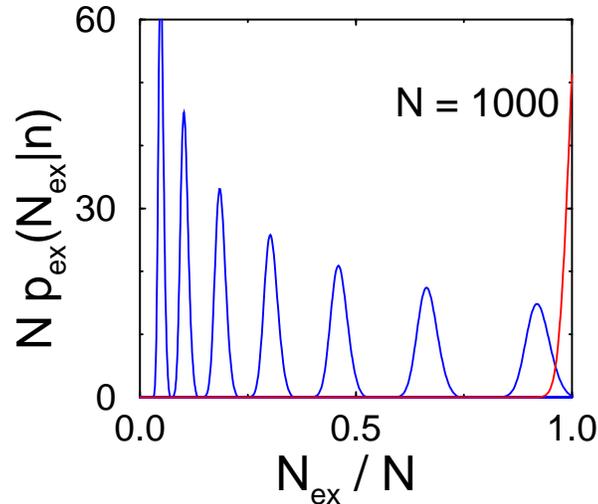


Fig. 2. Microcanonical probability distributions  $p_{\text{ex}}(N_{\text{ex}}|n)$  for finding  $N_{\text{ex}}$  out of  $N = 1000$  ideal Bose particles, trapped by a three-dimensional isotropic harmonic potential, excited when the total excitation energy  $E$  is  $n \cdot \hbar\omega$ , with  $\omega$  denoting the oscillator frequency. The number  $n$  determines the temperature  $T$ . The normalized temperatures  $T/T_0$  corresponding to the blue, Gaussian-like distributions range from 0.3 to 0.9 (left to right, in steps of 0.1);  $T_0 = (\hbar\omega/k_B)(N/\zeta(3))^{1/3}$ . Due to finite- $N$ -effects, the condensation temperature is lowered from  $T_0$  to about  $0.93 T_0$ . The temperature corresponding to the rightmost, red distribution is  $T = 0.95 T_0$ , lying slightly above the condensation point.

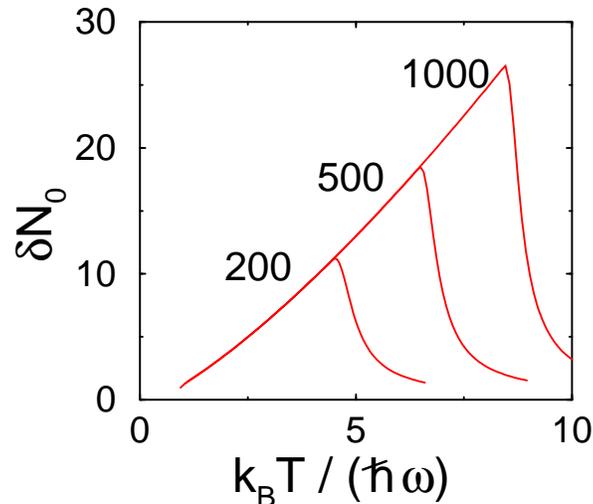


Fig. 3. Microcanonical fluctuations  $\delta N_0$  for  $N = 200, 500,$  and  $1000$  ideal Bose particles trapped by a three-dimensional, isotropic harmonic potential. The fluctuations are maximal close to the respective condensation points. These maximal fluctuations scale approximately as  $\sqrt{N}$ , cf. Eqs. (18) and (25). Note that the low-temperature fluctuations for all three systems agree perfectly, thus demonstrating the  $N$ -independence of  $\delta N_0$  below the condensation point.

Having thus gained confidence in the abilities of Maxwell's Demon, we now set it to work in order to compute condensate fluctuations  $\delta N_0$ . To this end, we consider ideal Bose gases in  $d$ -dimensional traps with arbitrary single-particle energies  $\varepsilon_\nu$ ; we stipulate  $\varepsilon_0 = 0$ . Denoting the grand canonical partition function by  $\Xi(z, \beta)$ , we base our analysis on its "excited" part  $\Xi_{\text{ex}}(z, \beta) \equiv (1 - z)\Xi(z, \beta)$ . Since, by virtue of Eq. (7),

$$\Xi_{\text{ex}}(z, \beta) = \sum_{N_{\text{ex}}=0}^{\infty} z^{N_{\text{ex}}} \sum_E \Phi(N_{\text{ex}}|E) e^{-\beta E}, \quad (8)$$

this function has the decisive property

$$\left( z \frac{\partial}{\partial z} \right)^k \Xi_{\text{ex}}(z, \beta) \Big|_{z=1} = \sum_E \left( \sum_{N_{\text{ex}}=0}^{\infty} N_{\text{ex}}^k \Phi(N_{\text{ex}}|E) \right) e^{-\beta E} \equiv M_k(\beta), \quad (9)$$

i.e., it yields directly the non-normalized *canonical* moments  $M_k(\beta)$ , and generates the *microcanonical* moments

$$\mu_k(E) \equiv \sum_{N_{\text{ex}}=0}^{\infty} N_{\text{ex}}^k \Phi(N_{\text{ex}}|E) \quad \text{with} \quad k = 0, 1, 2, \dots \quad (10)$$

We then employ the Maxwell's Demon approximation: as long as there is a condensate, these moments (with  $N_{\text{ex}}$  ranging from 0 to  $\infty$ ) approximate the true moments of the physical set  $\{ \Phi(N_{\text{ex}}|E) \}$  (where the number  $N_{\text{ex}}$  of excited particles can not exceed the total particle number  $N$ ); in this approximation one has the identity

$\mu_0(E) = \Omega(E|N)$  [15]. Now the calculations within the canonical ensemble become remarkably simple. The canonical expectation value  $\langle N_{\text{ex}} \rangle = N - \langle N_0 \rangle$  of the number of excited particles is given by

$$\langle N_{\text{ex}} \rangle = \frac{M_1(\beta)}{M_0(\beta)}; \quad (11)$$

the canonical condensate fluctuations  $(\delta N_0)_{cn}^2 = (\delta N_{\text{ex}})_{cn}^2$  follow from

$$(\delta N_0)_{cn}^2 = \frac{M_2(\beta)}{M_0(\beta)} - \left( \frac{M_1(\beta)}{M_0(\beta)} \right)^2. \quad (12)$$

Without any further approximation, these expressions can be rewritten as complex integrals:

$$\langle N_{\text{ex}} \rangle = \frac{1}{2\pi i} \int_{\tau-i\infty}^{\tau+i\infty} dt \Gamma(t) Z(\beta, t) \zeta(t) \quad (13)$$

and

$$(\delta N_0)_{cn}^2 = \frac{1}{2\pi i} \int_{\tau-i\infty}^{\tau+i\infty} dt \Gamma(t) Z(\beta, t) \zeta(t-1), \quad (14)$$

where  $\Gamma(t)$  and  $\zeta(t)$  denote the Gamma function and Riemann's Zeta function, respectively. All the information about the specific trap under consideration is embodied in its spectral Zeta function

$$Z(\beta, t) = \sum_{\nu=1}^{\infty} \frac{1}{(\beta \varepsilon_\nu)^t} \quad (15)$$

where the sum runs over the trap spectrum, excluding the ground state energy  $\varepsilon_0 = 0$ . The real number  $\tau$  in Eqs. (13) and (14) has to be chosen such that the path of integration up the complex  $t$ -plane sees all poles to its left.

So far, the analysis is quite general. We now specialize the further deliberations to ideal Bose gases in  $d$ -dimensional traps with power-law single-particle spectra

$$\varepsilon_{\{\nu_i\}} = \Delta \sum_{i=1}^d c_i \nu_i^\sigma, \quad \nu_i = 0, 1, 2, \dots, \quad \sigma > 0, \quad (16)$$

where the dimensionless coefficients  $c_i$  characterize the trap's anisotropy, normalized such that the lowest  $c_i$  is unity; the characteristic energy  $\Delta$  measures the gap between the ground state and the first excited state, and the exponent  $\sigma$  is determined by the potential's shape. Such systems have been studied first by de Groot *et al.* [19]; we have adopted here the notation also employed by Wilkens and Weiss [18].

If we consider  $N$ -asymptotically large systems and disregard finite- $N$ -effects, that is, if we focus on gases consisting of at least some  $10^5$  particles, say, then a good approximation to the density of states is provided by

$$\rho(E) = \frac{A}{\Gamma(\frac{d}{\sigma})} \left(\frac{E}{\Delta}\right)^{d/\sigma-1} \frac{1}{\Delta} \quad \text{with} \quad A \equiv \frac{\Gamma(\frac{1}{\sigma} + 1)^d}{\left(\prod_{i=1}^d c_i\right)^{1/\sigma}}. \quad (17)$$

Using this density, and assuming that the anisotropy coefficients  $c_i$  are not too different from each other [10], the usual line of reasoning shows that for  $d/\sigma > 1$  there is a sharp onset of Bose-Einstein condensation at the temperature  $T_0$  given by [19]

$$\frac{k_B T_0}{\Delta} = \frac{1}{A^{\sigma/d}} \left(\frac{N}{\zeta(\frac{d}{\sigma})}\right)^{\sigma/d}. \quad (18)$$

Moreover, the spectral Zeta functions can now be well approximated by

$$Z(\beta, t) \approx \frac{A}{\Gamma(\frac{d}{\sigma})} (\beta\Delta)^{-t} \zeta(t + 1 - d/\sigma), \quad (19)$$

so Eqs. (13) and (14) adopt the transparent forms

$$\langle N_{\text{ex}} \rangle \approx \frac{A}{\Gamma(\frac{d}{\sigma})} \frac{1}{2\pi i} \int_{\tau-i\infty}^{\tau+i\infty} dt (\beta\Delta)^{-t} \Gamma(t) \zeta(t + 1 - d/\sigma) \zeta(t) \quad (20)$$

and

$$(\delta N_0)_{cn}^2 \approx \frac{A}{\Gamma(\frac{d}{\sigma})} \frac{1}{2\pi i} \int_{\tau-i\infty}^{\tau+i\infty} dt (\beta\Delta)^{-t} \Gamma(t) \zeta(t + 1 - d/\sigma) \zeta(t - 1). \quad (21)$$

For  $\Delta \ll k_B T$ , the behavior of either integral is determined by the pole of its integrand farthest to the right in the complex plane. Keeping in mind that  $\zeta(z)$  has merely one single pole at  $z = 1$ , with residue 1, while the poles of  $\Gamma(z)$  are located at  $z = 0, -1, -2, \dots$ , the decisive pole is provided *either* by the system's Zeta function  $\zeta(t + 1 - d/\sigma)$ , *or* by the other Zeta function that is determined by the order of the cumulant one is asking for: by  $\zeta(t)$ , if one asks for the first cumulant  $\langle N_{\text{ex}} \rangle$ , or by  $\zeta(t - 1)$ , if one asks for the second cumulant  $(\delta N_0)^2$ . To see what the argument boils down to, let us first consider the evaluation of Eq. (20), where the system's pole at  $t = d/\sigma$  competes with the cumulant-order pole at  $t = 1$ :

- If  $d/\sigma > 1$ , the low-temperature behavior of  $\langle N_{\text{ex}} \rangle$  is governed by the pole of  $\zeta(t + 1 - d/\sigma)$  at  $t = d/\sigma$ . Hence the residue theorem yields

$$\langle N_{\text{ex}} \rangle \approx A \zeta(d/\sigma) \left( \frac{k_B T}{\Delta} \right)^{d/\sigma}. \quad (22)$$

This canonical result, valid for  $T < T_0$ , coincides precisely with the result of the customary grand canonical analysis [19]. For example, in the case of the three-dimensional isotropic harmonic oscillator potential (i.e., for  $d = 3$ ,  $\sigma = 1$ ,  $A = 1$  and  $\Delta = \hbar\omega$ ) Eq. (22) yields the familiar formula

$$\langle N_0 \rangle = N - \langle N_{\text{ex}} \rangle = N \left[ 1 - \left( \frac{T}{T_0} \right)^3 \right] \quad \text{for} \quad T < T_0 = \frac{\hbar\omega}{k_B} \left( \frac{N}{\zeta(3)} \right)^{1/3}.$$

- If  $d/\sigma = 1$ , both Zeta functions in Eq. (20) coincide. We then encounter a double pole at  $t = 1$ , and find

$$\langle N_{\text{ex}} \rangle \approx A \frac{k_B T}{\Delta} \left[ \ln \left( \frac{k_B T}{\Delta} \right) + \gamma \right], \quad (23)$$

where  $\gamma = 0.5772\dots$  is Euler's constant. This corresponds to a result obtained already in 1950 by Nanda [20] with the help of the Euler-Maclaurin summation formula.

- If  $0 < d/\sigma < 1$ , the pole of  $\zeta(t)$  at  $t = 1$  takes over:

$$\langle N_{\text{ex}} \rangle \approx \frac{A}{\Gamma(\frac{d}{\sigma})} \zeta(2 - d/\sigma) \frac{k_B T}{\Delta}, \quad (24)$$

so that for sufficiently low temperatures  $\langle N_{\text{ex}} \rangle$  now depends linearly on  $T$ , regardless of the value of  $d/\sigma$  that characterizes the trap.

A mere glance at Eq. (21) then suffices to reveal that *the very same scenario* — a first pole at  $t = d/\sigma$  that endows the temperature dependence with a trap-specific exponent as long as it lies to the right of a second one, which yields universal behavior when it becomes dominant — also governs the canonical condensate fluctuations, with the only difference that the second pole now is located at  $t = 2$ :

- If  $d/\sigma > 2$ , the pole of  $\zeta(t + 1 - d/\sigma)$  at  $t = d/\sigma$  wins, giving

$$(\delta N_0)_{cn}^2 \approx A \zeta(d/\sigma - 1) \left( \frac{k_B T}{\Delta} \right)^{d/\sigma}. \quad (25)$$

- If  $d/\sigma = 2$ , we find at  $t = 2$  the already familiar double pole, resulting in

$$(\delta N_0)_{cn}^2 \approx A \left( \frac{k_B T}{\Delta} \right)^2 \left[ \ln \left( \frac{k_B T}{\Delta} \right) + \gamma + 1 \right]. \quad (26)$$

- If  $0 < d/\sigma < 2$ , the pole of  $\zeta(t - 1)$  at  $t = 2$  lies to the right of its rival, yielding

$$(\delta N_0)_{cn}^2 \approx \frac{A}{\Gamma(\frac{d}{\sigma})} \zeta(3 - d/\sigma) \left( \frac{k_B T}{\Delta} \right)^2. \quad (27)$$

In particular, for the one-dimensional harmonic oscillator we have  $d = 1$ ,  $\sigma = 1$ ,  $A = 1$  and  $\Delta = \hbar\omega$ , so that we recover our previous microcanonical result (2) within the canonical ensemble, recalling that  $\zeta(2) = \pi^2/6$ .

The above canonical fluctuations, derived from the integral representation (14), reduce to the expression obtained by Politzer [21] in the case of the three-dimensional isotropic trap, and match the results obtained by Wilkens and Weiss [18].

The calculation of the corresponding *microcanonical* quantities now requires saddle-point inversions of Eq. (9) in order to obtain the microcanonical moments  $\mu_k(E)$  from the canonical moments  $M_k(\beta)$ . Performing these inversions, and reexpressing energy in terms of temperature, we find that the integral (13) — and, hence, the results (22), (23), and (24) for the number of excited particles — remains valid within the microcanonical ensemble. The fluctuations require more care: Whereas canonical and microcanonical fluctuations coincide in the large- $N$ -limit for  $d/\sigma < 2$ , the microcanonical mean-square fluctuations  $(\delta N_0)_{mc}^2$  are distinctly lower than their canonical counterparts for  $d/\sigma > 2$ :

$$(\delta N_0)_{cn}^2 - (\delta N_0)_{mc}^2 \approx \frac{Ad}{d + \sigma} \frac{\zeta^2\left(\frac{d}{\sigma}\right)}{\zeta\left(\frac{d}{\sigma} + 1\right)} \left(\frac{k_B T}{\Delta}\right)^{d/\sigma} \quad \text{for } d/\sigma > 2 \text{ and } T < T_0. \quad (28)$$

Thus, the exponent of  $T$  is the same for both  $(\delta N_0)_{cn}^2$  and  $(\delta N_0)_{mc}^2$ , but the prefactors can differ substantially. This Eq. (28) contains as a special case the result obtained for the three-dimensional isotropic trap by Navez *et al.* [14].

Before summarizing these findings, it is useful to also consider the heat capacities for trapped ideal Bose gases with the single-particle spectra (16): for  $d/\sigma > 1$ , and temperatures below the condensation temperature  $T_0$ , the heat capacity per particle is given by

$$\frac{C_{<}}{Nk_B} = \frac{d}{\sigma} \left(\frac{d}{\sigma} + 1\right) \frac{\zeta\left(\frac{d}{\sigma} + 1\right)}{\zeta\left(\frac{d}{\sigma}\right)} \left(\frac{T}{T_0}\right)^{d/\sigma}, \quad (29)$$

above  $T_0$  by

$$\frac{C_{>}}{Nk_B} = \frac{d}{\sigma} \left(\frac{d}{\sigma} + 1\right) \frac{g_{d/\sigma+1}(z)}{g_{d/\sigma}(z)} - \frac{d^2}{\sigma^2} \frac{g_{d/\sigma}(z)}{g_{d/\sigma-1}(z)}. \quad (30)$$

Since the fugacity  $z$  approaches unity from below when  $T$  approaches  $T_0$  from above, so that the Bose function  $g_\alpha(z)$  approaches  $\zeta(\alpha)$ , we see that the heat capacity remains continuous at  $T_0$  for  $0 < d/\sigma \leq 2$ , but exhibits a jump of size

$$\left. \frac{C_{<} - C_{>}}{Nk_B} \right|_{T_0} = \frac{d^2}{\sigma^2} \frac{\zeta\left(\frac{d}{\sigma}\right)}{\zeta\left(\frac{d}{\sigma} - 1\right)} \quad (31)$$

for  $d/\sigma > 2$ .

We thus arrive at the following picture: For any dimension  $d$  and trap exponent  $\sigma > 0$ , the fluctuation of the number of condensate particles is independent of the total particle number  $N$ . For isolated traps, this insensitivity of the system with respect to  $N$  reflects the well-peakedness of the microcanonical distributions (4), see Fig. 2: if there is a condensate, the behavior of the ideal Bose gas does not depend on how many particles the condensate consists of. If  $d/\sigma < 2$ , so that the heat capacity remains continuous in the large- $N$ -limit, canonical and microcanonical fluctuations  $\delta N_0$  vanish linearly with temperature, see Eq. (27). If  $d/\sigma = 2$ , there appears a logarithmic correction to the linear  $T$ -dependence, as quantified by Eq. (26). But if  $d/\sigma > 2$ , so that the heat capacity becomes discontinuous, then the fluctuations  $\delta N_0$  vanish proportionally to  $T^{d/2\sigma}$ , so that now the properties of the trap determine the way the fluctuations depend on temperature. In addition, in this case the microcanonical fluctuations are markedly lower than the fluctuations in a trap that exchanges energy with a heat bath.

Intuitively, one might have expected some sort of square root law for the fluctuations. Because of the  $N$ -independence of the condensate fluctuations, there is, of course, no “ $\sqrt{N}$ -dependence” of  $\delta N_0$ . The square root is hidden elsewhere: Since  $\delta N_0 = \delta N_{\text{ex}}$  for ensembles with fixed particle number  $N$ , we find

- $\delta N_{\text{ex}} \propto \langle N_{\text{ex}} \rangle^{1/2}$  for  $2 < d/\sigma$ ;
- $\delta N_{\text{ex}} \propto \langle N_{\text{ex}} \rangle^{\sigma/d}$  for  $1 < d/\sigma < 2$ ;
- $\delta N_{\text{ex}} \propto \langle N_{\text{ex}} \rangle$  for  $0 < d/\sigma < 1$ ,

with proportionality constants that are independent of temperature, both canonically and microcanonically. The first of these relations is just what one might have guessed, but the crossover from normal fluctuations for  $d/\sigma > 2$  to much stronger fluctuations for  $0 < d/\sigma < 1$  appears noteworthy.

It remains to be seen how much of this ideal structure survives in the case of weakly interacting Bose gases. It should also be recognized that Maxwell’s Demon, though it has provided the microcanonical low-temperature fluctuations, can not solve all problems of the ideal gas. When considering  $d$ -dimensional isotropic harmonic traps, the Maxwell’s Demon approximation (i.e., the replacement of the true upper summation bound “ $N$ ” in Eq. (10) by “ $\infty$ ”) is *exact* below the “restriction temperature” (i.e., that temperature where the number  $n = E/(\hbar\omega)$  of energy quanta equals the number  $N$  of particles [15]), but the description of the Bose–Einstein transition itself is beyond the capabilities of Maxwell’s Demon. Namely, that description requires the computation of the numbers  $\Omega(n\hbar\omega|N)$  of microstates also under conditions where the restriction due to the finite  $N$  becomes decisive. Incidentally, one meets the task of computing such restricted partitions of integers also in other problems of statistical mechanics, for example in the theory of the so-called compact lattice animals, or of the infinite-state Potts model [22].

Nonetheless, the results obtained with the help of the Maxwell’s Demon approximation have some interesting number-theoretical implications. Going once more back to Eq. (2) for the one-dimensional oscillator, and inserting the energy–temperature relation  $n = E/(\hbar\omega) \approx \zeta(2)(k_B T/\hbar\omega)^2$ , we find the truly remarkable formula

$$\delta N_0 \approx \sqrt{n} . \tag{32}$$

This has a twofold interpretation. The physicist, puzzled by the loss of the square root fluctuation law at the level of  $\langle N_{\text{ex}} \rangle$ , finds a substitute:

- *For ideal Bose particles trapped at low temperatures by a one-dimensional harmonic potential, the root-mean-square fluctuation of the number of ground state particles is given by the square root of the number of energy quanta.*

The mathematician, who approaches Eq. (32) from the viewpoint of partition theory, sees the solution to another problem:

- *If one considers all unrestricted partitions of the integer  $n$  into positive, integer summands, and asks for the root-mean-square fluctuation of the number of summands, then the answer is (asymptotically) just  $\sqrt{n}$*

— certainly one of the most amazing examples for the occurrence of square root fluctuations! The ease with which the solution to a seemingly difficult number-theoretical question has been obtained here is even aesthetically appealing. It is pleasing to conclude that ongoing developments in statistical mechanics, themselves being motivated by recent experimental achievements [6,7,8], have a high potential for further fertilization across subfield boundaries.