Invisible cloak design with controlled constitutive parameters and arbitrary shaped boundaries through Helmholtz's equation

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Abstract: An approach to design an invisible cloak with controlled constitutive parameters and arbitrary shaped boundaries is presented. Helmholtz's equation is adopted to establish a mapping between original and transformed coordinates inside the cloak. Then the constitutive parameters are obtained by the established mapping. The analytical solution of a regular cloak and the numerical solution of an irregular cloak both verify that that our method will guide electromagnetic wave efficiently and control the constitutive parameters of the cloak conveniently. It has great significance in realizing a cloak practically.

OCIS codes: (230.3205) Invisibility cloaks; (160.1190) Anisotropic optical materials; (230.0230) Optical devices.

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1. Introduction

Since the theory of invisible cloak was firstly presented in 2006 [1-2], this realm has got much attention. The basic principle of this theory is to keep Maxwell’s equations form-invariant, when the original coordinate system has altered to a new topology. A distorted space with a hollow region can be created by coordinate transformation. If the constitutive parameters have changed accordingly, the EM wave will propagate around the hollow region smoothly. This phenomenon has been verified experimentally by Schurig et al. [3] in 2006 and the cloak they realized is cylindrical-form. Cloaks with more complicated shapes have been studied gradually. Axes symmetrical cloaks, such as polygonal [4-5] and elliptical [6-8] shapes have been designed by analytical functions. In recent months, various methods on achieving arbitrary shaped cloaks have been proposed. For instance, Yan et al [9] gives a method based on analyzing the wave vectors on cloak boundaries. Li et al [10] and Ma et al [11-12] distort polar axes into interior region of the cloak to realize a compressed space. Jiang et al [13] and Nicolet et al. [14] respectively use different particular functions to describe the irregular boundaries. Hu et al. [15] solves Laplace’s equation to get the transformed coordinates inside cloak. Although those processes can guide EM wave efficiently, they are not flexible to realize a parameter controllable cloak with arbitrary shapes. Nonlinear transformation is valid to control the variation of coordinates of the cloak with regular shape [16], but once utilized to the cloak with arbitrary shape, this approach will cause a few inconvenience in programming. The methods given by [9-15] are utilized to realize an arbitrary cloak, but due to the fixed boundary conditions, the transformed coordinates can not be altered, which means the material parameters are fixed or uncontrollable. It is inconvenient when those constitutive parameters are not desired or hard to implement practically.

Toward implement a material parameters controllable electromagnetic cloak with arbitrary shaped boundaries, this paper proposes utilizing Helmholtz’s equation to get the relationship between original and transformed coordinates inside the cloak. The constitutive parameters can be easily obtained by the Jacobian transformation matrix. Full wave simulation results indicate that by using this method, the electromagnetic wave will propagate around the obstacle shielded by the cloak with no scattering. The constitutive parameters of the cloak are verified being controlled by the propagation constant of Helmholtz’s equation.

2. Helmholtz’s equation method for parameter controllable cloak

According to the theory of coordinate transformation [1], if the original coordinates (x_1, x_2, x_3) are distorted to the new coordinates $(x_1'(x_1, x_2, x_3), x_2'(x_1, x_2, x_3), x_3'(x_1, x_2, x_3))$, in order to keep Maxwell’s equations form-invariant, the relationship between original constitutive parameters and transformed constitutive parameters must satisfy the following expression

$$\varepsilon_r^{ij} = A_{i'}^{i} A_{j'}^{j} \varepsilon_r^{i'j'} [\det(\mathbf{A})]^{-1} \tag{1}$$

$$\mu_r^{ij} = A_{i'}^{i} A_{j'}^{j} \mu_r^{i'j'} [\det(\mathbf{A})]^{-1} \tag{2}$$

where ε_r^{ij} and μ_r^{ij} are the permittivity and permeability in original space respectively, $\varepsilon_r^{i'j'}$ and $\mu_r^{i'j'}$ are those two in distorted space, $A_{i'}^{i} = \partial x_i / \partial x_i'$ is the gradient coefficient of inverse mapping $x_i' \to x_i$, $\mathbf{A} = [A_{i'}^{i}]$ is the Jacobian transformation matrix. Rewrite Eqs. (1) and (2) as

$$\varepsilon_r^{i'j'} = (A_{i'}^{i})^{-1} (A_{j'}^{j})^{-1} \varepsilon_r^{ij} [\det(\mathbf{A})] \tag{3}$$

$$\mu_r^{i'j'} = (A_{i'}^{i})^{-1}(A_{j'}^{j})^{-1}\mu_r^{ij}[\det(\mathbf{A})] \tag{4}$$

where $(A_{i'}^{i})^{-1}$ is the element of inverse matrix $(\mathbf{A})^{-1}$. Figure 1 shows a section of irregular cloak. Region Ⅰ and Ⅱ respectively denote free space and a section of the cloak with arbitrary shaped boundary curves Γ_{outer} and Γ_{inner}. Region Ⅲ is shielded by the cloak, which can be seen as the expansion of point $P(x_1|_P, x_2|_P, x_3|_P)$.

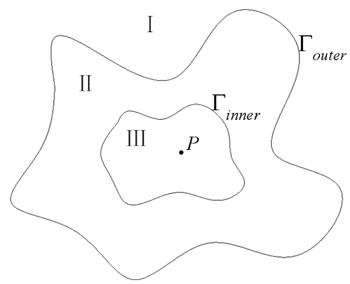

Fig. 1. Geometrical graph of the section of irregular invisible cloak

To ensure electromagnetic wave propagating into cloak with no reflection, the outer boundary condition should be $x_i|_{\Gamma_{outer}} = x_i'$. In region Ⅲ, no EM field exists, thus the inner boundary condition is chosen as $x_i|_{\Gamma_{inner}} = x_i|_P$. In region Ⅱ, we utilize homogeneous Helmholtz's equation and specified boundary conditions to describe the original coordinates as

$$\begin{cases} \left(\dfrac{\partial^2}{\partial x_1'^2} + \dfrac{\partial^2}{\partial x_2'^2} + \dfrac{\partial^2}{\partial x_3'^2}\right)x_i + k^2 x_i = 0 \\ x_i|_{\Gamma_{outer}} = x_i', \quad x_i|_{\Gamma_{inner}} = x_i|_P \end{cases} \tag{5}$$

where k is the propagation constant. From Eq. (5) we can obtain original coordinate x_i straightly. If boundary curves Γ_{outer} and Γ_{inner} are both regular, x_i will be given as an analytical function $x_i = f_i\ (x_1', x_2', x_3')$. If those boundaries are irregular, it's difficult to obtain the analytical solution of Eq. (5), and the original coordinate x_i will be solved numerically. If k equal to zero, Eq. (5) will change to a Laplace's equation. This situation has been discussed by Hu et al [15]. Insert x_i to Eqs. (3) and (4), we can get the permittivity and permeability in region Ⅱ. For a regular spherical cloak, the outer and inner boundaries are both spherical surface, the distortion is just in the radial direction, and Eq. (5) will be changed to the following form

$$\begin{cases} \dfrac{1}{r'^2}\left[\dfrac{\partial}{\partial r'}\left(r'^2\dfrac{\partial r}{\partial r'}\right)\right] + k^2 r = 0 \\ r|_{r'=r_1} = 0, \quad r|_{r'=r_2} = r_2 \end{cases} \tag{6}$$

where r_1, r_2 are the radius of the inner and outer boundary of the cloak respectively, the solution of Eq. (6) is given by

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$$r = A \cdot \frac{\sin kr'}{kr'} - B \cdot \frac{\cos kr'}{kr'} \tag{7}$$

with

$$A = \frac{kr_2^2 \cos kr_1}{\sin[k(r_2 - r_1)]}, \; B = \frac{kr_2^2 \sin kr_1}{\sin[k(r_2 - r_1)]} \tag{8}$$

where $k(r_2\text{-}r_1) \neq n\pi$, n is an arbitrary integer. Equation (7) indicates that the original coordinate r is not only relative to the transformed coordinate r' but also to the propagation constant k. Besides, the changing of r is nonlinear. Thus although boundary conditions are fixed, r is still alterable with k. That means the constitutive parameters of cloak will be controllable by the propagation constant k. We can obtain gradient coefficient as

$$A_{r'}^{r} = \frac{\partial r}{\partial r'} = A\left(\frac{\cos kr'}{r'} - \frac{\sin kr'}{kr'^2}\right) + B\left(\frac{\sin kr'}{r'} + \frac{\cos kr'}{kr'^2}\right) \tag{9}$$

Let $\theta = \theta'$, $\varphi = \varphi'$, the other transformation coefficient are

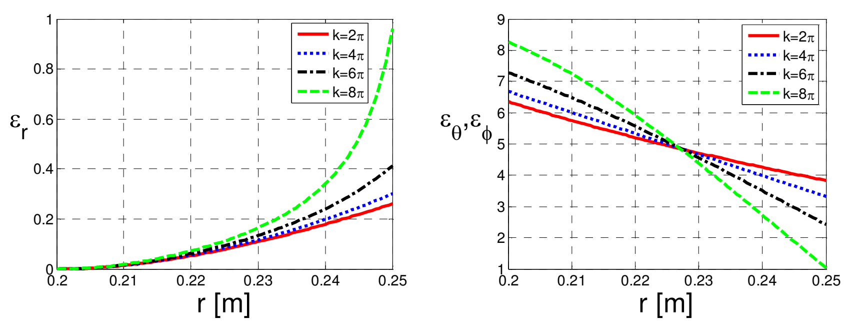
$$A_{\theta'}^{\theta} = A_{\varphi'}^{\varphi} = \frac{r}{r'} = A\frac{\sin kr'}{kr'^2} - B\frac{\cos kr'}{kr'^2} \tag{10}$$

From Eq. (3), (4), (9) and (10), the constitutive parameters of spherical cloak are

$$\varepsilon_r' = \mu_r' = \frac{\left(A\dfrac{\sin kr'}{kr'^2} - B\dfrac{\cos kr'}{kr'^2}\right)^2}{A\left(\dfrac{\cos kr'}{r'} - \dfrac{\sin kr'}{kr'^2}\right) + B\left(\dfrac{\sin kr'}{r'} + \dfrac{\cos kr'}{kr'^2}\right)} \tag{11}$$

$$\varepsilon_\theta' = \varepsilon_\varphi' = \mu_\theta' = \mu_\varphi' = A\left(\frac{\cos kr'}{r'} - \frac{\sin kr'}{kr'^2}\right) + B\left(\frac{\sin kr'}{r'} + \frac{\cos kr'}{kr'^2}\right) \tag{12}$$

Where A, B are given by Eq. (8). The curve clusters of Eq. (11) and Eq. (12) for different k are shown respectively in Fig. 2, where $r_1 = 0.2$ m, $r_2 = 0.25$ m.



(a) ε_r (b) ε_θ and ε_φ

Fig. 2. Curves of constitute parameters with variable k inside the cloak

Because the value of μ is equal to ε, we deal with ε alone in the next discussion. In Fig. 2(a), ε_r near r_2 varies obviously with different k. For $k < 6\pi$, ε_r increases steadily and the maximum of which is under 0.41. When $k = 8\pi$, the value of ε_r jumps intensely to 1 in the region between 0.24m and 0.25m. In Fig. 2(b), the varying range of ε_θ and ε_φ also rely on k. When k

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shifts from 2π to 8π, this range changes from [6.3, 3.8] to [8.2, 1]. Thus we can adopt different groups of constitutive parameters to form a cloak, which may give us a chance to choose appropriate parameters to realize a real cloak.

3. Designing a parameter controllable cloak with arbitrary boundaries

For the reason of simplifying, we choose a 2D-cloak shown in Fig. 1 as the model to make full-wave simulation by COMSOL Multi-physics Software, which is based on finite element method (FEM). In the simulation, we fill region Ⅲ with perfect electric conductor (PEC) and apply scattering boundary conditions to the outside of calculation area. Perfectly matched layers (PML) are used to absorb TM plane wave with z-polarized electric fields generated by E-field boundary. The frequency of plane wave is 7.5 GHz and the calculation area is set to be 0.48m×0.48m. Similar to the processing Hu et al. [15] adopted, we use PDE model of Helmholtz' s equation to calculate original coordinates in region Ⅱ firstly. The constitutive parameters of the cloak are obtained secondly. Then utilize TE wave model to verify the property of cloak we designed, this whole processing will compute 1704399 unknowns for each k. Figure 3 shows the distribution of electric field near the cloak with different k. Figure 3 indicates that in any case of k, TM wave will penetrate the cloak with very low refraction and propagate around the PEC region smoothly. When k increases from 10π to 50π, in the area near inner boundary, the electric field become more homogeneous, while in the region near outer boundary, the field seems been compressed and relies on the shape of cloak.

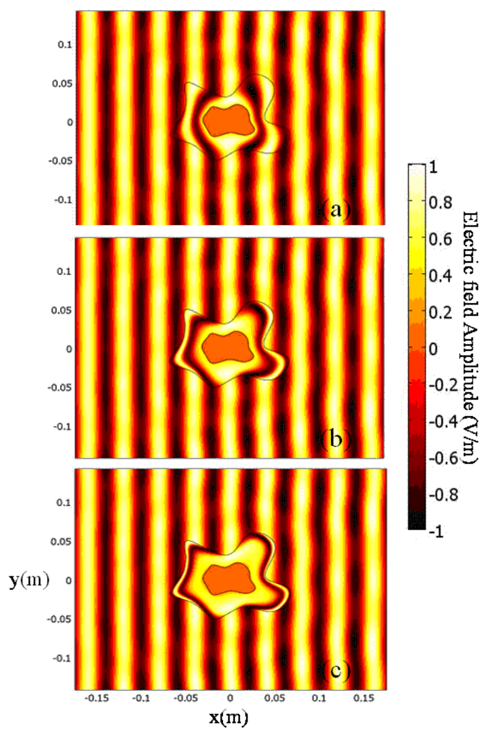

Fig. 3. Distribution of electric-field near the cloak with different k. (a) $k = 10\pi$. (b) $k = 30\pi$. (c) $k = 50\pi$.

Figure 4 displays the distribution of ε_{zz} with different k. As can be seen, the region of $\varepsilon_{zz} < 1$ expands from the space near inner boundary to the most area inside the cloak, when k increases from 10π to 50π. In the region near outer boundary, ε_{zz} increase rapidly to the value above 4. For k choose 50π, this value even equal to 10. The other constitutive parameters μ_{xx},

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μ_{yy} and μ_{zz} can be easily obtained and analyzed directly through the distribution figures similar to Fig. 4, which is not mentioned here. The above discussion has verified that the material parameters of the cloak are controlled by varying *k*. The electromagnetic cloak designed by Schurig et al [3] is constituted by 10 layers of split-ring resonators (SRRs). The relative permeability μ_r of this cloak increases from 0.003 to 0.279 and the maximum change to the SRRs structure is less than 0.09 mm. As the shift of μ_r relies on tiny alteration to the structure, the precision in fabricating SRRs should be strict. Utilizing our approach, the constitutive parameters will be chosen in a wide range for a certain *k*. It can effectively reduce the restriction of fabricating precision, especially in the case that the structure is very small. Thus it is convenient and flexible in accomplishing a real invisible cloak.

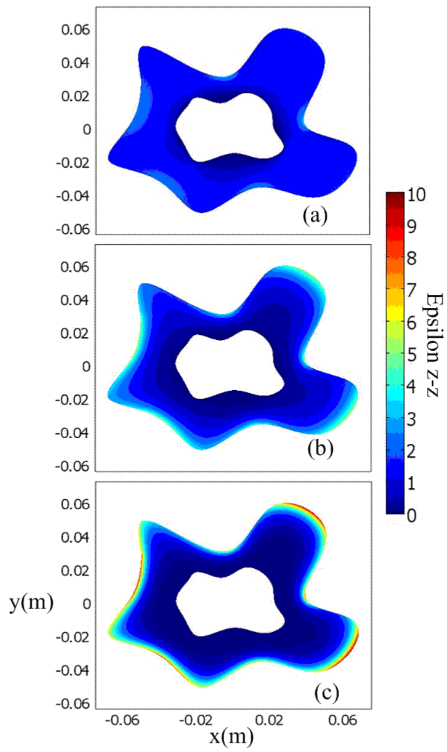

Fig. 4. Distribution of ε_{zz} inside the cloak with different *k*. (a) $k = 10\pi$. (b) $k = 30\pi$. (c) $k = 50\pi$.

4. Conclusion

A method using Helmholtz's equation to fabricate a parameter controllable cloak is demonstrated. The constitutive parameters of a regular spherical cloak have been obtained and verified to be controlled by the propagation constant of Helmholtz's equation. An arbitrary shaped 2D-cloak has been designed by our method and simulated by FEM. The results coincide with what we expect. Although the cloak simulated in this paper is given as 2D, our method is still usable in designing a 3D irregular shaped cloak. In conclusion, this approach can guide wave efficiently and control the constitutive parameters conveniently, which will make the designing of arbitrary shaped cloak more flexible. It has great significance in engineering and should be given more attention.

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